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DISTRIBUTION OF EIGENVALUES FOR THE DISCONTINUOUS BOUNDARY-VALUE PROBLEM WITH FUNCTIONAL-MANYPOINT CONDITIONS

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ABSTRACT

In this study, we investigate the boundary-value problem with eigenvalue parameter generated by the differential equation with discontinuous coefficients and boundary conditions which contains not only endpoints of the considered interval, but also a point of discontinuity, a finite number of internal points and abstract linear functionals. So our problem is not a pure boundary-value one.

We single out a class of linear functionals and find simple algebraic conditions on the coefficients which guarantee the existence of an infinite number of eigenvalues. Also, the asymptotic formulas for the eigenvalues are found.

The results obtained in this paper are new, even in the case of boundary conditions either without internal points or without linear functionals.

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1. Introduction

The investigation of boundary-value problems for which the eigenvalue parameter appears in both the equation and the boundary conditions originates from the works of G. D. Birkhoff [1, 2]. Many papers and books deal with the spectral properties of such problems (see, for example, [5, 8, 9, 10, 11, 12, 13] and the corresponding bibliography).

In many monographs and papers, the theory of boundary-value problems for ordinary differential equations is usually considered for equations with a constant coefficient at the highest derivative and for boundary conditions which contain only endpoints of the considered interval. However, this paper deals with one nonstandard boundary-value problem for a second-order ordinary differential equation with discontinuous coefficients and boundary conditions containing not only endpoints of the considered interval, but also a point of discontinuity, a finite number of internal points and abstract linear functionals. Moreover, the eigenvalue parameter appears in both the differential equation and the boundary conditions.

Some spectral properties of such problems and applications to the corresponding initial-boundary-value problems for parabolic equations were investigated by O. Sh. Mukhtarov [5] and by O. Sh. Mukhtarov and H. Demir [6].

Similar problems for differential equations with discontinuous coefficients, but without the internal points and abstract linear functionals in the boundary conditions, were investigated by M. L. Rasulov in monographs [8, 9].

One must note that in the series of S. and Y. Yakubov's works published in recent years, they have constructed an abstract theory of boundary-value problems with a parameter in the boundary conditions (see [11, 12, 13] and corresponding bibliography). In these works, in particular, the main spectral properties of the more general problems, but with continuous coefficients, are investigated.

2. Statement of the problem

Let us consider a differential equation

$$a(x)y'' + c(x)y = \lambda^2 y, \quad x \in [-1,0) \cup (0,1] \tag{1}$$

with the functional-manypoint boundary conditions

$$L_i(y) = \sum_{k=0}^{1} \lambda^{1-k}(a_{ik}y^{(k)}(-1) + \delta_{ik}y^{(k)}(-0) + \gamma_{ik}y^{(k)}(+0) + \beta_{ik}y^{(k)}(1)$$

$$(2) \qquad + \sum_{s=1}^{2} \sum_{p=1}^{n_{ik}^{s}} \eta_{ip}^{ks} y^{(k)}(\xi_{ip}^{ks}) + T_{ik} y) = 0, \quad i = 1,2,3,4,$$

where $a(x)$ and $c(x)$ are complex-valued functions; $a(x) = a_1$ at $x \in [-1, 0)$, $a(x) = a_2$ at $x \in (0, 1]$, and a_{ik}, δ_{ik}, γ_{ik}, β_{ik}, η_{ip}^{ks} are complex coefficients; $\xi_{ip}^{k1} \in (-1, 0)$ and $\xi_{ip}^{k2} \in (0, 1)$ are internal points; T_{vk} are abstract linear functionals; $y^{(k)}(\pm 0)$ denotes $\lim_{x \to \pm 0} y^{(k)}(x)$.

Below, $W_q^k(-1,0) + W_q^k(0,1)$, $q \in (1, \infty)$, $k = 0, 1, 2, \ldots$, denotes the Banach space of complex-valued functions $y = y(x)$ defined on $[-1, 0) \cup (0, 1]$, which belongs to $W_q^k(-1,0)$ and $W_q^k(0,1)$ on intervals $(-1,0)$ and $(0,1)$, respectively, with norm $\|y\|_{q,k} = (\|y\|^q_{W_q^k(-1,0)} + \|y\|^q_{W_q^k(0,1)})^{1/q}$, where $W_q^k(-1,0)$ and $W_q^k(0,1)$ are the usual Sobelev spaces.

Note that, without loss of generality, we consider equation (1) instead of the more general equation

$$a(x)y'' + b(x)y' + c(x)y = \lambda^2 y,$$

since, by using the substitution $y = \tilde{y}e^{\phi(x)}$, where

$$\phi(x) = \begin{cases} -\frac{1}{2a_1} \int_{-1}^{x} b(t)y(t) & \text{at } x \in [-1, 0), \\ -\frac{1}{2a_2} \int_{0}^{x} b(t)y(t) & \text{at } x \in (0, 1], \end{cases}$$

we find that this equation takes the form (1) with the same eigenvalue parameter λ. Also, it is easy to verify that under this substitution the form of boundary conditions (2) is not changed. Besides, for simplicity we replace the general domain $[a, c) \cup (c, b]$, $a < c < b$, by $[-1, 0) \cup (0, 1]$, since we can return to the general case by making the substitution

$$x = \begin{cases} c + (c-a)t & \text{at } t \in [-1, 0), \\ c + (b-c)t & \text{at } t \in (0, 1). \end{cases}$$

Some special cases of problem (1)–(2) arise in various physical transfer problems, in particular, in heat and mass transfer problems [3].

3. Eigenvalues of the problem

As usual, those values of parameter λ for which the considered boundary-value problem (1)–(2) has a nontrivial solution are called the eigenvalues of the problem (1)–(2).

Let $y_{10}(x,\lambda)$, $y_{20}(x,\lambda)$ and $y_{30}(x,\lambda)$, $y_{40}(x,\lambda)$ denote some fundamental systems of solutions of the differential equation (1) on $[-1,0)$ and $(0,1]$, respectively. By defining

$$y_j(x,\lambda) = \begin{cases} y_{j0}(x,\lambda), & x \in I_n \\ 0, & x \notin I_n \end{cases} \tag{3}$$

where $I_1 = I_2 = [-1,0)$ and $I_3 = I_4 = (0,1]$, the general solution of equation (1) can be represented in the form

$$y(x,\lambda) = \sum_{j=1}^{4} c_j y_j(x,\lambda). \tag{4}$$

Substituting (4) into boundary conditions (2) yields a system of linear homogeneous equations

$$\sum_{j=1}^{4} c_j L_v(y_j) = 0, \quad v = 1,2,3,4 \tag{5}$$

for the determination of the constants $c_j, j = 1,\ldots,4$. Consequently, the eigenvalues of the problem (1)–(2) consist of the zeros of the characteristic determinant

$$\Delta(\lambda) = \det(L_v y_j)_{v,j=1,\ldots,4}. \tag{6}$$

First, for the considered problem, we shall divide the complex λ-plane into specific sectors in which, in turn, we shall find the asymptotic expressions for solutions of the differential equation, for boundary functionals and boundary-value forms. Then, by substituting these obtained asymptotic expressions into the equation $\Delta(\lambda) = 0$ we shall find the corresponding asymptotic formulas for the eigenvalues. Note that such formulas are not only of interest in themselves, but they may also be used to establish the complete and basic properties of the system of eigenfunctions and associated functions of the considered problem.

The cases $\arg a_1 \neq \arg a_2$ and $\arg a_1 = \arg a_2$ are examined separately.

4. Asymptotic behaviour of eigenvalues for the case $\arg a_1 \neq \arg a_2$

4.1. SEPARATION OF THE COMPLEX λ-PLANE INTO SPECIFIC SECTORS. Throughout the paper we employ the notation $\omega_1 = (\sqrt{a_1})^{-1}$, $\omega_2 = -(\sqrt{a_1})^{-1}$, $\omega_3 = (\sqrt{a_2})^{-1}$, $\omega_4 = -(\sqrt{a_2})^{-1}$, where $\sqrt{z} := |z|e^{i(\arg z)/2}$, $-\pi < \arg z \leq \pi$. Divide the complex λ-plane into four sectors $S_v, v = 1,\ldots,4$, by the rays $l_k = \{\lambda \in \mathbb{C} |\ \operatorname{Re}\lambda\omega_k = 0, (-1)^k \operatorname{Im}\lambda\omega_k \leq 0\}$. On all of these sectors each of the real-valued functions $\operatorname{Re}\lambda\omega_k$ is of a single sign, since these functions can vanish only on boundaries of the sectors S_v. Let us consider one of the sectors (S_v) with fixed index v. Using the same considerations as in [7] it is easy to verify that for equation (1) there exists a fundamental system of particular solutions $y_{10}(x,\lambda)$, $y_{20}(x,\lambda)$ on $[-1,0)$ and $y_{30}(x,\lambda)$, $y_{40}(x,\lambda)$ on $(0,1]$, respectively, which are regular analytic functions of $\lambda \in S_v$ for sufficiently large $|\lambda|$, and which with their derivatives can be expressed in the asymptotic form

$$(7) \qquad \begin{cases} y_{k0} = e^{\lambda\omega_k x}(1+O(1/\lambda)), \\ y'_{k0} = \lambda\omega_k e^{\lambda\omega_k x}(1+O(1/\lambda)). \end{cases}$$

Here, as usual, the expression $O(1/\lambda)$ denotes any function of the form $f(x,\lambda)/\lambda$, where $|f(x,\lambda)|$ always remains less than a constant for $x \in I_k$ and sufficiently large $|\lambda|$.

Now let l'_j $(j = 1,\dots,4)$ be arbitrary rays, originating from the point $\lambda = 0$ and differing from l_j, and situated so as to form the sequence

$$(8) \qquad l_1, l'_1, l_3, l'_3, l_2, l'_2, l_4, l'_4.$$

The rays l'_j divide each sector S_v into two subsectors. Thus we have eight sectors which we shall denote as Ω_j, $j = 1,\dots,8$.

As seen from the construction, the sectors $\Omega = \{\Omega_1,\Omega_2,\dots,\Omega_8\}$ can be distributed into two groups, $\Omega^{(1)} = \{\Omega_1^{(1)},\dots,\Omega_4^{(1)}\}$ and $\Omega^{(2)} = \{\Omega_1^{(2)},\dots,\Omega_4^{(2)}\}$, such that the group $\Omega^{(k)}$ includes those sectors Ω_j in which $\operatorname{Re}\lambda(\sqrt{a_k})^{-1} \to \infty$ as $\lambda \to \infty$.

4.2. Asymptotic expressions for the characteristic determinant $\Delta(\lambda)$ in the Ω sectors for large values of $|\lambda|$. Each of the real-valued functions $\operatorname{Re}\lambda\omega_k$ does not change its sign also in every sector Ω_j, since each of them is a subsector of some sector S_v.

Let $y_k = y_k(x,\lambda)$, $k = 1,\dots,4$, be functions defined as (3) for the solutions $y_{k0}(x,\lambda)$ of equation (1) in I_k, for which the asymptotic expressions (7) are satisfied.

First we need to estimate asymptotically the expressions $T_{vk}y_n(.,\lambda)$ as $\lambda \to \infty$ in the sectors Ω_j. Since the linear functionals T_{vk} acts from $W_q^k(-1,0)+W_q^k(0,1)$ into the complex plane $\mathbb{C}$ continuously, in virtue of the general representation of the continuous linear functionals in the $L_q(a,b)$ spaces and using well-known methods of real analysis, it may be shown that there exists a function $U_{vks} \in$

$W_p^k(-1,0)+W_p^k(0,1)$ such that the equalities

$$T_{vk}(y)=\sum_{s=0}^{k}\left(\int_{-1}^{0}y^{(s)}(x)U_{vks}(x)dx+\int_{0}^{1}y^{(s)}(x)U_{vks}(x)dx\right), \quad v=1,\ldots,4,\ k=0,1, \tag{9}$$

hold for every $y\in W_q^k(-1,0)+W_q^k(0,1)$, where $1/p+1/q=1$ [4].

In only one of the sectors of the group $\Omega^{(1)}$ do the relations $\operatorname{Re}\lambda\omega_3\geq 0$ and $\operatorname{Re}\lambda\omega_1\to+\infty$ as $\lambda\to\infty$ hold, and in only one of the sectors of the group $\Omega^{(2)}$ do the relations $\operatorname{Re}\lambda\omega_1\geq 0$ and $\operatorname{Re}\lambda\omega_3\to+\infty$ as $\lambda\to\infty$ hold. We shall denote these sectors as $\Omega_0^{(1)}$ and $\Omega_0^{(2)}$, respectively. We shall calculate asymptotic approximation expressions for $T_{vk}y_j$ only for these sectors of the groups $\Omega^{(1)}$ and $\Omega^{(2)}$, since calculations for others sectors can be made analogously. For convenience below, by $[M]$, $M\in\mathbb{C}$, we denote any sum of the form $M+f(\lambda)$, when $f(\lambda)\to 0$ as $\lambda\to\infty$.

First let λ vary in $\Omega_0^{(1)}$. Substituting (7) into (9), remembering that $\omega_2=-\omega_1$, $\omega_4=-\omega_3$ and applying the well-known Riemann–Lebesgue Lemma [cf. 8, p. 117, Lemma 7], we get

$$\begin{aligned}
T_{vk}(y_1)&=\sum_{s=0}^{k}\int_{-1}^{0}y_{10}^{(s)}(x,\lambda)U_{vks}(x)dx\\
&=\sum_{s=0}^{k}\int_{-1}^{0}(\lambda\omega_1)^s e^{\lambda\omega_1 x}(1+O(1/\lambda))U_{vks}(x)dx\\
&=(\lambda\omega_1)^k\sum_{s=0}^{k}(\lambda\omega_1)^{s-k}\int_{0}^{1}e^{-\lambda\omega_1 x}\{U_{vks}(-x)(1+O(1/\lambda))\}dx\\
&=\lambda^k[0], && (10)\\
T_{vk}(y_2)&=\sum_{s=0}^{k}\int_{-1}^{0}y_{20}^{(s)}(x,\lambda)U_{vks}(x)dx\\
&=\sum_{s=0}^{k}(\lambda\omega_2)^s e^{-\lambda\omega_2}\int_{01}^{0}e^{\lambda\omega_2(1+x)}(1+O(1/\lambda))U_{vks}(x)dx\\
&=(\lambda\omega_2)^k e^{-\lambda\omega_2}\sum_{s=0}^{k}(\lambda\omega_2)^{s-k}\int_{0}^{1}e^{-\lambda\omega_2 x}\{U_{vks}(x-1)(1+O(1/\lambda))\}dx\\
&=\lambda^k e^{-\lambda\omega_2}[0], && (11)\\
T_{vk}(y_3)&=\sum_{s=0}^{k}\int_{0}^{1}y_{30}^{(s)}(x-\lambda)U_{vks}(x)dx
\end{aligned}$$

$$= \sum_{s=0}^{k} (\lambda\omega_3)^s e^{\lambda\omega_3} \int_0^1 e^{-\lambda\omega_3(1-x)}(1+O(1/\lambda))U_{vks}(x)dx$$

$$= (\lambda\omega_3)^k e^{\lambda\omega_3} \sum_{s=0}^{k} (\lambda\omega_3)^{s-k} \int_0^1 e^{-\lambda\omega_3 x}\{U_{vks}(1-x)(1+O(1/\lambda))\}dx$$

$$= \lambda^k e^{\lambda\omega_3}[0], \tag{12}$$

$$T_{vk}(y_4) = \sum_{s=0}^{k} \int_0^1 y_{40}^{(s)}(x,\lambda)U_{vks}(x)dx$$

$$= \sum_{s=0}^{k} \int_0^1 (\lambda\omega_4)^s e^{\lambda\omega_4 x}(1+O(1/\lambda))U_{vks}(x)dx$$

$$= (\lambda\omega_4)^k \sum_{s=0}^{k} (\lambda\omega_4)^{s-k} \int_0^1 e^{-\lambda\omega_3 x}\{U_{vks}(x)(1+O(1/\lambda))\}dx$$

$$= \lambda^k[0]. \tag{13}$$

Using these formulas we find the next asymptotic expressions for $L_v(y_j)$ in $\Omega_0^{(1)}$, as $\lambda \to \infty$:

$$L_i(y_1) = \sum_{k=0}^{1} \lambda^{1-k}\{\alpha_{ik}(\lambda\omega_1)^k e^{-\lambda\omega_1}[1] + \delta_{ik}(\lambda\omega_1)^k[1]$$

$$+ \sum_{p=1}^{n_{ik}^1} \eta_{ip}^{k1}(\lambda\omega_1)^k e^{\lambda\omega_1\xi_{ip}^{k1}}[1] + (\lambda\omega_1)^k[0]\}$$

$$=\lambda[\delta_{i0} + \omega_1\delta_{i1}], \tag{14}$$

$$L_i(y_2) = \sum_{k=0}^{1} \lambda^{1-k}\{\alpha_{ik}(\lambda\omega_2)^k e^{-\lambda\omega_2}[1]$$

$$+ \delta_{ik}(\lambda\omega_2)^k[1] + \sum_{p=1}^{n_{ik}^1} \eta_{ip}^{k1}(\lambda\omega_2)^k e^{\lambda\omega_2\xi_{ip}^{k1}}[1] + (\lambda\omega_2)^k e^{-\lambda\omega_2}[0]\}$$

$$=\lambda e^{-\lambda\omega_2} \sum_{k=0}^{1} \{[a_{ik}\omega_2^k] + [\delta_{ik}\omega_2^k]e^{\lambda\omega_2} + \sum_{p=1}^{n_{ik}^1}[\eta_{ip}^{k1}\omega_2^k]e^{\lambda\omega_2(\xi_{ip}^{k1}+1)}\}$$

$$=\lambda e^{-\lambda\omega_2}[\alpha_{i0} + \omega_2\alpha_{i1}], \tag{15}$$

$$L_i(y_3) = \sum_{k=0}^{1} \lambda^{1-k}\{\gamma_{ik}(\lambda\omega_3)^k[1] + \beta_{ik}(\lambda\omega_3)^k e^{\lambda\omega_3}[1]$$

$$+ \sum_{p=1}^{n_{ik}^2} \eta_{ip}^{k2}(\lambda\omega_3)^k e^{\lambda\omega_3\xi_{ip}^{k2}}[1] + (\lambda\omega_3)^k e^{\lambda\omega_3}[0]\}$$

$$(16)\qquad =\lambda\{[\gamma_{i0}+\omega_3\gamma_{i1}]+e^{\lambda\omega_3}[\beta_{i0}+\omega_3\beta_{i1}]+\sum_{k=0}^{1}\sum_{p=1}^{n_{ik}^2}e^{\lambda\omega_3\xi_{ip}^{k2}}[\omega_e^k\eta_{ip}^{k2}]\}$$

and analogously

$$(17)\qquad L_i(y_4)=\lambda\{[\gamma_{i0}+\omega_4\gamma_{i1}]+e^{\lambda\omega_4}[\beta_{i0}+\omega_4\beta_{i1}]+\sum_{k=0}^{1}\sum_{p=1}^{n_{ik}^2}e^{\lambda\omega_4\xi_{ip}^{k2}}[\omega_4\eta_{ip}^{k2}]\}$$

for $i=1,2,3,4$.

All calculations for the sector $\Omega_0^{(2)}$ are carried out analogously. After further calculations we get the next asymptotic expressions,

$$(18)\qquad \begin{cases} T_{vk}(y_1)=\lambda^k[0], & T_{vk}(y_w)=\lambda^k e^{-\lambda\omega_2}[0],\\ T_{vk}(y_e)=\lambda^k e^{\lambda\omega_3}[0], & T_{vk}(y_4)=\lambda^k[0], \end{cases}$$

and

(19)

$$\begin{cases} L_i(y_j)=\lambda\{[\alpha_{i0}+\omega_j\alpha_{i1}]e^{-\lambda\omega_j}+[\delta_{i0}+\omega_j\delta_{i1}]\\ \qquad\qquad +\sum_{k=0}^{1}\sum_{p=1}^{n_{ik}^1}[\omega_j^k\eta_{ip}^{k1}]e^{\lambda\omega_j\xi_{ip}^{k1}}\}, \quad j=1,2,\\ L_i(y_3)=\lambda e^{\lambda\omega_3}[\beta_{i0}+\omega_3\beta_{i1}],\\ L_i(y_4)=\lambda[\gamma_{i0}+\omega_4\lambda_{i1}], \end{cases}$$

in the sector $\Omega_0^{(2)}$ as $\lambda\to\infty$.

By substituting the asymptotic expressions (14)–(17) for $L_i(y_j)$ into the determinant $\Delta(\lambda)$, taking the common factors λ of each column and the common factor $e^{-\lambda\omega_2}$ of the second column outside the determinant, and taking into account $\omega_2=-\omega_1$, $\omega_4=-\omega_3$, we may represent this determinant in the asymptotic form

$$(20)\qquad \Delta(\lambda)=\lambda^4e^{\lambda\omega_1}([A_1^{(1)}]e^{m_1\lambda\omega_3}+[A_2^{(1)}]e^{m_2\lambda\omega_3}+\cdots+[A_{\sigma_1}^{(1)}]e^{m_{\sigma_1}\lambda\omega_3}),$$

where $-1=m_1<m_2<\cdots<m_{\sigma_1}=1$; $A_j^{(k)}$ are some complex numbers.

Furthermore, it is easy to see that

$$A_1^{(1)}=\begin{vmatrix} \delta_{10}+\omega_1\delta_{11} & \alpha_{10}+\omega_2\alpha_{11} & \gamma_{10}+\omega_3\gamma_{11} & \beta_{10}+\omega_4\beta_{11}\\ \delta_{20}+\omega_1\delta_{21} & \alpha_{20}+\omega_2\alpha_{21} & \gamma_{20}+\omega_3\gamma_{21} & \beta_{20}+\omega_4\beta_{21}\\ \delta_{30}+\omega_1\delta_{31} & \alpha_{30}+\omega_2\alpha_{31} & \gamma_{30}+\omega_3\gamma_{31} & \beta_{30}+\omega_4\beta_{31}\\ \delta_{40}+\omega_1\delta_{41} & \alpha_{40}+\omega_2\alpha_{41} & \gamma_{40}+\omega_3\gamma_{41} & \beta_{40}+\omega_4\beta_{41} \end{vmatrix}$$

and

$$A_{\sigma_1}^{(1)}=\begin{vmatrix} \delta_{10}+\omega_1\delta_{11} & \alpha_{10}+\omega_2\alpha_{11} & \beta_{10}+\omega_4\beta_{11} & \gamma_{10}+\omega_3\gamma_{11}\\ \delta_{20}+\omega_1\delta_{21} & \alpha_{20}+\omega_2\alpha_{21} & \beta_{20}+\omega_4\beta_{21} & \gamma_{20}+\omega_3\gamma_{21}\\ \delta_{30}+\omega_1\delta_{31} & \alpha_{30}+\omega_2\alpha_{31} & \beta_{30}+\omega_4\beta_{31} & \gamma_{30}+\omega_3\gamma_{31}\\ \delta_{40}+\omega_1\delta_{41} & \alpha_{40}+\omega_2\alpha_{41} & \beta_{40}+\omega_4\beta_{41} & \gamma_{40}+\omega_3\gamma_{41} \end{vmatrix}.$$

It can be shown similarly that the characteristic determinant $\Delta(\lambda)$ in the sector $\Omega_0^{(2)}$ has the asymptotic representation

$$\Delta(\lambda) = \lambda^4 e^{\lambda\omega_3}([A_1^{(2)}]e^{n_1\lambda\omega_1} + [A_2^{(2)}]e^{n_2\lambda\omega_1} + \cdots + [A_{\sigma_2}^{(2)}]e^{n_{\sigma_2}\lambda\omega_1}) \tag{21}$$

where $-1 = n_1 < n_2 < \cdots < n_{\sigma_2} = 1$,

$$A_1^{(2)} = \begin{vmatrix} \alpha_{10} + \omega_2\alpha_{11} & \delta_{10} + \omega_1\delta_{11} & \beta_{10} + \omega_4\beta_{11} & \gamma_{10} + \omega_3\gamma_{11} \\ \alpha_{20} + \omega_2\alpha_{21} & \delta_{20} + \omega_1\delta_{21} & \beta_{20} + \omega_4\beta_{21} & \gamma_{20} + \omega_3\gamma_{21} \\ \alpha_{30} + \omega_2\alpha_{31} & \delta_{30} + \omega_1\delta_{31} & \beta_{30} + \omega_4\beta_{31} & \gamma_{30} + \omega_3\gamma_{31} \\ \alpha_{40} + \omega_2\alpha_{41} & \delta_{40} + \omega_1\delta_{41} & \beta_{40} + \omega_4\beta_{41} & \gamma_{40} + \omega_3\gamma_{41} \end{vmatrix}$$

and

$$A_{\sigma_2}^{(2)} = \begin{vmatrix} \delta_{10} + \omega_1\delta_{11} & \alpha_{10} + \omega_2\alpha_{11} & \beta_{10} + \omega_4\beta_{11} & \gamma_{10} + \omega_3\gamma_{11} \\ \delta_{20} + \omega_1\delta_{21} & \alpha_{20} + \omega_2\alpha_{21} & \beta_{20} + \omega_4\beta_{21} & \gamma_{20} + \omega_3\gamma_{21} \\ \delta_{30} + \omega_1\delta_{31} & \alpha_{30} + \omega_2\alpha_{31} & \beta_{30} + \omega_4\beta_{31} & \gamma_{30} + \omega_3\gamma_{31} \\ \delta_{40} + \omega_1\delta_{41} & \alpha_{40} + \omega_2\alpha_{41} & \beta_{40} + \omega_4\beta_{41} & \gamma_{40} + \omega_3\gamma_{41} \end{vmatrix}.$$

4.3. Asymptotic behaviour of the eigenvalues. Now for the case $\arg a_1 \neq \arg a_2$ we can find the asymptotic formulas for the eigenvalues of the considered problem.

Theorem 1: *Let the following conditions be satisfied:*

(1) $\arg a_1 \neq \arg a_2$;

(2) $c(\cdot) \in L_q(-1,1)$, $q > 1$;

(3)

$$\theta_{ij} = \begin{vmatrix} \alpha_{10}\sqrt{a_1}+(-1)^i\alpha_{11} & \delta_{10}\sqrt{a_1}+(-1)^{i+1}\delta_{11} & \beta_{10}\sqrt{a_2}+(-1)^j\beta_{11} & \gamma_{10}\sqrt{a_2}+(-1)^{j+1}\gamma_{11} \\ \alpha_{20}\sqrt{a_1}+(-1)^i\alpha_{21} & \delta_{20}\sqrt{a_1}+(-1)^{i+1}\delta_{21} & \beta_{20}\sqrt{a_2}+(-1)^j\beta_{21} & \gamma_{20}\sqrt{a_2}+(-1)^{j+1}\gamma_{21} \\ \alpha_{30}\sqrt{a_1}+(-1)^i\alpha_{31} & \delta_{30}\sqrt{a_1}+(-1)^{i+1}\delta_{31} & \beta_{30}\sqrt{a_2}+(-1)^j\beta_{31} & \gamma_{30}\sqrt{a_2}+(-1)^{j+1}\gamma_{31} \\ \alpha_{40}\sqrt{a_1}+(-1)^i\alpha_{41} & \delta_{40}\sqrt{a_1}+(-1)^{i+1}\delta_{41} & \beta_{40}\sqrt{a_2}+(-1)^j\beta_{41} & \gamma_{40}\sqrt{a_2}+(-1)^{j+1}\gamma_{41} \end{vmatrix}$$

$\neq 0$,

$i = 1,2$, $j = 1,2$;

(4) *the linear functionals T_{vk} in the spaces $W_q^k(-1,0)+W_q^k(0,1)$ are continuous.*

Then the boundary-value problem (1)–(2), has in each sector S_v a precise number of eigenvalues whose asymptotic behaviour may be expressed by the following formulas:

$$\lambda_n^{(1)} = \sqrt{a_1}\pi(1 + O(1/n)), \tag{22}$$

$$\lambda_n^{(2)} = -\sqrt{a_1}\pi(1 + O(1/n)), \tag{23}$$

$$\lambda_n^{(3)} = \sqrt{a_2}\pi(1 + O(1/n)), \tag{24}$$

$$\lambda_n^{(4)} = -\sqrt{a_2}\pi(1 + O(1/n)), \quad n = 1,2,\ldots. \tag{25}$$

Proof: The rays l'_j (for $j = 1, \dots, 4$) divide the complex λ-plane into four sectors $R_1, \dots, R_4$. Let R_j be that sector which contains the ray l_j. We shall distribute these sectors into two groups $R^{(1)} = \{R_1, R_2\}$ and $R^{(2)} = \{R_3, R_4\}$. Obviously, each sector of the group $R^{(k)}$ consists of two sectors of the groups $\Omega^{(k)}$. By $R_0^{(k)}$ we denote that sector of the group $R^{(k)}$ which contains $\Omega_0^{(k)} (k = 1, 2)$.

As seen from the considerations in subsections 4.1 and 4.2, the asymptotic expressions (20) and (21) hold also in sectors $R_0^{(1)}$ and $R_0^{(2)}$, respectively. Let $R_1^{(1)}$ and $R_1^{(2)}$ be the other sectors of the groups $R^{(1)}$ and $R^{(2)}$, respectively.

In a similar way to subsections 4.1 and 4.2, one can prove that the characteristic determinant $\Delta(\lambda)$ has the asymptotic representations given by

$$\Delta(\lambda) = \lambda^4 e^{-\lambda\omega_1}([B_1^{(1)}]e^{s_1\lambda\omega_3} + [B_2^{(1)}]e^{s_2\lambda\omega_3} + \cdots + [B_{\tau_1}^{(1)}]e^{s_{\tau_1}\lambda\omega_3}) \tag{26}$$

and

$$\Delta(\lambda) = \lambda^4 e^{-\lambda\omega_3}([B_1^{(2)}]e^{t_1\lambda\omega_1} + [B_2^{(2)}]e^{t_2\lambda\omega_1} + \cdots + [B_{\tau_2}^{(2)}]e^{t_{\tau_2}\lambda\omega_1}) \tag{27}$$

in the sectors $R_1^{(1)}$ and $R_1^{(2)}$, respectively, where

$$-1 = s_1 < s_2 < \cdots < s_{\tau_1} = 1, -1 = t_1 < t_2 < \cdots < t_{\tau_2} = 1,$$

$$B_1^{(1)} = \begin{vmatrix} \alpha_{10} + \omega_1\alpha_{11} & \delta_{10} + \omega_2\delta_{11} & \gamma_{10} + \omega_3\gamma_{11} & \beta_{10} + \omega_4\beta_{11} \\ \alpha_{20} + \omega_1\alpha_{21} & \delta_{20} + \omega_2\delta_{21} & \gamma_{20} + \omega_3\gamma_{21} & \beta_{20} + \omega_4\beta_{21} \\ \alpha_{30} + \omega_1\alpha_{31} & \delta_{30} + \omega_2\delta_{31} & \gamma_{30} + \omega_3\gamma_{31} & \beta_{30} + \omega_4\beta_{31} \\ \alpha_{40} + \omega_1\alpha_{41} & \delta_{40} + \omega_2\delta_{41} & \gamma_{40} + \omega_3\gamma_{41} & \beta_{40} + \omega_4\beta_{41} \end{vmatrix},$$

$$B_{\tau_1}^{(1)} = \begin{vmatrix} \alpha_{10} + \omega_1\alpha_{11} & \delta_{10} + \omega_2\delta_{11} & \beta_{10} + \omega_3\beta_{11} & \gamma_{10} + \omega_4\gamma_{11} \\ \alpha_{20} + \omega_1\alpha_{21} & \delta_{20} + \omega_2\delta_{21} & \beta_{20} + \omega_3\beta_{21} & \gamma_{20} + \omega_4\gamma_{21} \\ \alpha_{30} + \omega_1\alpha_{31} & \delta_{30} + \omega_2\delta_{31} & \beta_{30} + \omega_3\beta_{31} & \gamma_{30} + \omega_4\gamma_{31} \\ \alpha_{40} + \omega_1\alpha_{41} & \delta_{40} + \omega_2\delta_{41} & \beta_{40} + \omega_3\beta_{41} & \gamma_{40} + \omega_4\gamma_{41} \end{vmatrix},$$

$$B_1^{(2)} = \begin{vmatrix} \alpha_{10} + \omega_1\alpha_{11} & \delta_{10} + \omega_2\delta_{11} & \gamma_{10} + \omega_3\gamma_{11} & \beta_{10} + \omega_4\beta_{11} \\ \alpha_{20} + \omega_1\alpha_{21} & \delta_{20} + \omega_2\delta_{21} & \gamma_{20} + \omega_3\gamma_{21} & \beta_{20} + \omega_4\beta_{21} \\ \alpha_{30} + \omega_1\alpha_{31} & \delta_{30} + \omega_2\delta_{31} & \gamma_{30} + \omega_3\gamma_{31} & \beta_{30} + \omega_4\beta_{31} \\ \alpha_{40} + \omega_1\alpha_{41} & \delta_{40} + \omega_2\delta_{41} & \gamma_{40} + \omega_3\gamma_{41} & \beta_{40} + \omega_4\beta_{41} \end{vmatrix},$$

$$B_{\tau_2}^{(2)} = \begin{vmatrix} \delta_{10} + \omega_1\delta_{11} & \alpha_{10} + \omega_2\alpha_{11} & \gamma_{10} + \omega_3\gamma_{11} & \beta_{10} + \omega_4\beta_{11} \\ \delta_{20} + \omega_1\delta_{21} & \alpha_{20} + \omega_2\alpha_{21} & \gamma_{20} + \omega_3\gamma_{21} & \beta_{20} + \omega_4\beta_{21} \\ \delta_{30} + \omega_1\delta_{31} & \alpha_{30} + \omega_2\alpha_{31} & \gamma_{30} + \omega_3\gamma_{31} & \beta_{30} + \omega_4\beta_{31} \\ \delta_{40} + \omega_1\delta_{41} & \alpha_{40} + \omega_2\alpha_{41} & \gamma_{40} + \omega_3\gamma_{41} & \beta_{40} + \omega_4\beta_{41} \end{vmatrix}.$$

According to condition (3) of Theorem 1, the principal terms of the first and last coefficients of the asymptotic quasi-polynomials (20), (21), (26) and (27), namely the numbers $A_1^{(1)}$, $A_{\sigma_1}^{(1)}$, $A_1^{(2)}$, $A_{\sigma_2}^{(2)}$, $B_1^{(1)}$, $B_{\tau_1}^{(1)}$, $B_1^{(2)}$, $B_{\tau_2}^{(2)}$, are different from zero.

Since $\Delta(\lambda) = \Delta_i^{(j)}(\lambda)$ when λ varies in sector $R_i^{(j)}$ and all quasi-polynomials $\Delta_i^{(j)}(\lambda)$ have the same form, it is enough to investigate only one of them. Namely, we shall investigate the equation $\Delta(\lambda) = 0$ only in the sector $R_0^{(1)}$, i.e., the equation

$$[A_1^{(1)}]e^{m_1\lambda\omega_3} + \cdots + [A_{\sigma_1}^{(1)}]e^{m_{\sigma_1}\lambda\omega_3} = 0. \tag{28}$$

By virtue of [8, p. 100, Lemma 1], equation (28) has in sector $R_0^{(1)}$ an infinite number of roots λ_n which are contained in a strip

$$\Pi_0^{(1)} = \{\lambda \in \mathbb{C} | |\operatorname{Re} \lambda\omega_3| < h/2\}$$

of finite width $h > 0$ and have the asymptotic expression

$$|\lambda_n\omega_3| = |\pi n(1 + O(1/n))|. \tag{29}$$

Taking into account that $\lambda_n \in \Pi_0^{(1)}$ and $\lambda_n \in R_0^{(1)}$, from (29) we get the sought asymptotic formula,

$$\lambda_n = \sqrt{a_2}\pi(1 + O(1/n)), \quad n = \pm 1, \pm 2, \ldots, \tag{30}$$

where there is only one possible choice for the sign of the integer n.

By the same consideration as used for sector $R_0^{(1)}$, it can be shown that the asymptotic behaviour of the eigenvalues contained in the sector $R_1^{(1)}$ also has the same form as (30), but the integer n has the opposite sign. The other formulas (22) and (23) can be obtained by the same procedure we used in proving formula (30).

5. Asymptotic behaviour of eigenvalues for the case $\arg a_1 = \arg a_2$

5.1. Separation of the complex λ-plane into half-planes. In the case $\arg a_1 = \arg a_2$ the lines $l_1 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_1 = 0\}$ and $l_2 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_3 = 0\}$ coincide, so the line $l = l_1 = l_2$ divides the complex λ-plane into two half-planes. In each of these half-planes the real-valued functions $\operatorname{Re}(\lambda\omega_1)$ and $\operatorname{Re}(\lambda\omega_3)$ do not change their signs and, moreover, have the same sign. These half-planes we denote as

$$\sum\nolimits_1 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_1 \geq 0, \operatorname{Re} \lambda\omega_3 \geq 0\}$$

and

$$\sum\nolimits_2 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_1 \geq 0, \operatorname{Re} \lambda\omega_3 \geq 0\}.$$

5.2. Asymptotic expressions for the characteristic determinant $\Delta(\lambda)$ in the half-planes $\sum_1$ and $\sum_2$. First, we find a suitable asymptotic representation of $T_{vk}(y_n)$ when $\lambda \to \infty$ in the half-planes $\sum_1$ and $\sum_2$, where $y_n = y_n(x, \lambda)$ are the same solutions as in section 4.2.

By similar considerations used in the case $\arg a_1 \neq \arg a_2$, it can be shown that for $T_{ik}(y_j)$ we have the asymptotic representations

$$(31) \qquad \begin{cases} T_{ik}(y_1) = \lambda^k[0], & T_{ik}(y_2) = \lambda^k e^{-\lambda\omega_2}[0], \\ T_{ik}(y_3) = \lambda^k e^{\lambda\omega_3}[0], & T_{ik}(y_4) = \lambda^k[0], \end{cases}$$

when $\lambda \in \sum_1$, $\lambda \to \infty$ and

$$(32) \qquad \begin{cases} T_{ik}(y_1) = \lambda^k e^{-\lambda\omega_1}[0], & T_{ik}(y_2) = \lambda^k[0], \\ T_{ik}(y_3) = \lambda^k[0], & T_{ik}(y_4) = \lambda^k e^{\lambda\omega_4}[0], \end{cases}$$

when $\lambda \in \sum_2$, $\lambda \to \infty$.

Using these expressions for $T_{ik}(y_j)$ we obtain the asymptotic representations

$$L_i(y_j) = \lambda([\alpha_{i0} + \omega_j\alpha_{i1}]e^{-\lambda\omega_j} + [\delta_{i0} + \omega_j\delta_{i1}] + \sum_{k=0}^{1}\sum_{p=1}^{n_{ip}^1}[\eta_{ip}^{k1}\omega_j^k]e^{\lambda\omega_j\xi_{ip}^{k1}}),$$
$$i = 1, \ldots, 4, j = 1, 2;$$

$$L_i(y_j) = \lambda([\gamma_{i0} + \omega_j\gamma_{i1}] + [\beta_{i0} + w_j\beta_{i1}]e^{\lambda\omega_j} + \sum_{k=0}^{1}\sum_{p=1}^{n_{ip}^2}[\eta_{ip}^{k2}\omega_j^k]e^{\lambda\omega_j\xi_{ip}^{k2}},$$
$$i = 1, \ldots, 4, j = 3, 4.$$

These are valid in both $\sum_1$ and $\sum_2$. Substituting these formulas into the characteristic determinant $\Delta(\lambda) = \det(L_i y_j)$, after some simple rearrangement we obtain the asymptotic representation

$$(33) \qquad \Delta(\lambda) = \lambda^4([P_1]e^{k_1\lambda(\omega_1+\omega_3)} + [P_2]e^{k_2\lambda(\omega_1+\omega_3)} + \cdots + [P_l]e^{k_l\lambda(\omega_1+\omega_3)}),$$

where $-1 = k_1 < k_2 < \cdots < k_l = 1$,

$$P_1 = \begin{vmatrix} \alpha_{10} + \omega_1\alpha_{11} & \delta_{10} + \omega_2\delta_{11} & \gamma_{10} + \omega_3\gamma_{11} & \beta_{10} + \omega_4\beta_{11} \\ \alpha_{20} + \omega_1\alpha_{21} & \delta_{20} + \omega_2\delta_{21} & \gamma_{20} + \omega_3\gamma_{21} & \beta_{20} + \omega_4\beta_{21} \\ \alpha_{30} + \omega_1\alpha_{31} & \delta_{30} + \omega_2\delta_{31} & \gamma_{30} + \omega_3\gamma_{31} & \beta_{30} + \omega_4\beta_{31} \\ \alpha_{40} + \omega_1\alpha_{41} & \delta_{40} + \omega_2\delta_{41} & \gamma_{40} + \omega_3\gamma_{41} & \beta_{40} + \omega_4\beta_{41} \end{vmatrix}$$

and

$$P_l = \begin{vmatrix} \alpha_{10} + \omega_1\alpha_{11} & \delta_{10} + \omega_2\delta_{11} & \beta_{10} + \omega_3\beta_{11} & \gamma_{10} + \omega_4\gamma_{11} \\ \alpha_{20} + \omega_1\alpha_{21} & \delta_{20} + \omega_2\delta_{21} & \beta_{20} + \omega_3\beta_{21} & \gamma_{20} + \omega_4\gamma_{21} \\ \alpha_{30} + \omega_1\alpha_{31} & \delta_{30} + \omega_2\delta_{31} & \beta_{30} + \omega_3\beta_{31} & \gamma_{30} + \omega_4\gamma_{31} \\ \alpha_{40} + \omega_1\alpha_{41} & \delta_{40} + \omega_2\delta_{41} & \beta_{40} + \omega_3\beta_{41} & \gamma_{40} + \omega_4\gamma_{41} \end{vmatrix}.$$

5.3. Asymptotic behaviour of the eigenvalues. Now we can prove the next theorem.

Theorem 2: *Let the following conditions be satisfied:*

(1) $\arg a_1 = \arg a_2$;
(2) $c(\cdot) \in L_q(-1,1)$, $q > 1$;
(3) $\theta_{12} \neq 0$, $\theta_{21} \neq 0$ *(θ_{12} and θ_{21} are the same determinant as in Theorem 1);*
(4) *the linear functionals T_{ik} in the spaces $W_q^k(-1,0)+W_q^k(0,1)$ are continuous* $(i = 1,\dots,4,\ k = 0,1)$.

Then the boundary-value problem (1)–(2) has in each half-plane $\sum_1$ and $\sum_2$ a precise number of eigenvalues whose asymptotic behaviour may be expressed by the following formulas:

$$\lambda_n^{(1)} = \frac{\sqrt{a_1}\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\pi ni(1+O(1/n)), \quad n = 1,2,\dots,$$
$$\lambda_n^{(2)} = \frac{\sqrt{a_1}\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\pi ni(1+O(1/n)), \quad n = 1,2,\dots.$$

Proof: According to condition (3) the principal terms of the first and last coefficients of the asymptotic quasi-polynomial (33), i.e., the numbers P_1 and P_l, are different from zero. Again, by virtue of [8, p. 100, Lemma 1] the quasi-polynomial (33) in the half-planes $\sum_1$ and $\sum_2$ has an infinite number of roots $\{\lambda_n^{(1)}\}$ and $\{\lambda_n^{(2)}\}$, respectively, which are contained in a strip

$$\Pi = \{\lambda \in \mathbb{C} |\ \operatorname{Re} \lambda(\omega_1+\omega_3) < h/2\}$$

of finite width $h > 0$ and have the asymptotic representations

$$|\lambda_n^{(s)}(\omega_1+\omega_3)| = |\pi n(1+O(1/n))|, \quad s = 1,2. \tag{34}$$

Taking into account that the eigenvalues $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ belong to the strip Π from (34), we get the sought asymptotic formulas:

$$\lambda_n^{(s)} = \frac{\sqrt{a_1}\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\pi ni(1+O(1/n)), \quad n = \pm 1, \pm 2, \dots, \ s = 1,2,$$

where there is only one possible choice for the sign of integer n, but opposite signs for the half-planes $\sum_1$ and $\sum_2$, which completes the proof.

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